Chapter 1
On sensitive dependence on initial conditions and existence of physical measure for 3-flows

Vítor Araújo

Abstract After reviewing known results on sensitiveness and also on robustness of attractors from Mañé and Morales, Pacifico and Pujals, together with observations on their proofs, we show that for attractors of three-dimensional flows, robust chaotic behavior (in sense of sensitiveness to initial conditions for the past as well for the future for all nearby flows) is equivalent to the existence of certain hyperbolic structures. These structures, in turn, are associated to the existence of physical measures. In short *in low dimensions, robust chaotic behavior for flows ensures the existence of a physical measure*.

Key words: sensitive dependence on initial conditions, physical measure, singular-hyperbolicity, expansiveness, robust attractor, robust chaotic flow, positive Lyapunov exponent

1.1 Introduction

The development of the theory of dynamical systems has shown that models involving expressions as simple as quadratic polynomials (as the *logistic family* or *Hénon attractor*, see e.g.[7] for a gentle introduction), or autonomous ordinary differential equations with a hyperbolic equilibrium of saddle-type accumulated by regular orbits, as the *Lorenz flow* (see e.g. [11, 29, 2]), exhibit *sensitive dependence on initial conditions*, a common feature of *chaotic dynamics*: small initial differences are rapidly augmented as time passes, causing two trajectories originally coming

Vítor Araújo
Instituto de Matemática, Universidade Federal do Rio de Janeiro,
C. P. 68.530, 21.945-970, Rio de Janeiro, RJ-Brazil
and
Centro de Matemática da Universidade do Porto,
Rua do Campo Alegre 687, 4169-007 Porto, Portugal
e-mail: : vitor.araujo@im.ufrj.br and vdaraujo@fc.up.pt

from practically indistinguishable points to behave in a completely different manner after a short while. Long term predictions based on such models are unfeasible since it is not possible to both specify initial conditions with arbitrary accuracy and numerically calculate with arbitrary precision. For an introduction to these notions see [7, 27].

Formally the definition of sensitivity is as follows for a flow X^t on some compact manifold M: a X^t-invariant subset Λ is *sensitive to initial conditions* or has *sensitive dependence on initial conditions*, or simply *chaotic* if, for every small enough $r > 0$ and $x \in \Lambda$, and for any neighborhood U of x, there exists $y \in U$ and $t \neq 0$ such that $X^t(y)$ and $X^t(x)$ are r-apart from each other: $\operatorname{dist}\big(X^t(y), X^t(x)\big) \geq r$. See Figure 1.1. An analogous definition holds for diffeomorphism f of some manifold, taking $t \in \mathbb{Z}$ and setting $f = X^1$ in the previous definition.

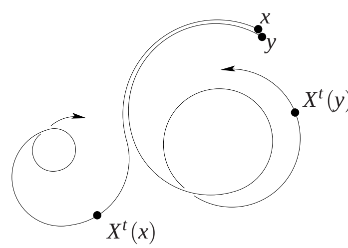

Fig. 1.1 Sensitive dependence on initial conditions.

Using some known results on robustness of attractors from Mañé [17] and Morales, Pacifico and Pujals [20] together with observations on their proofs, we show that for attractors of three-dimensional flows, robust chaotic behavior (in the above sense of sensitiveness to initial conditions) is equivalent to the existence of certain hyperbolic structures. These structures, in turn, are associated to the existence of physical measures. In short *in low dimensions, robust chaotic behavior ensures the existence of a physical measure*.

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1.2 Preliminary notions

Here and throughout the text we assume that M is a three-dimensional compact connected manifold without boundary endowed with some Riemannian metric which induces a distance denoted by dist and a volume form Leb which we name *Lebesgue*

measure or *volume*. For any subset A of M we denote by $\overline{A}$ the (topological) closure of A.

We denote by $\mathfrak{X}^r(M), r \geq 1$ the set of C^r smooth vector fields X on M endowed with the C^r topology. Given $X \in \mathfrak{X}^r(M)$ we denote by X^t, with $t \in \mathbb{R}$, the flow generated by the vector field X. Since we assume that M is a compact manifold the flow is defined for all time. Recall that the flow $(X^t)_{t\in\mathbb{R}}$ is a family of C^r diffeomorphisms satisfying the following properties:

1. $X^0 = Id : M \to M$ is the identity map of M;
2. $X^{t+s} = X^t \circ X^s$ for all $t, s \in \mathbb{R}$,

and it is *generated by the vector field* X if

3 $\frac{d}{dt}X^t(q)\big|_{t=t_0} = X\big(X_{t_0}(q)\big)$ for all $q \in M$ and $t_0 \in \mathbb{R}$.

We say that a compact X^t-invariant set Λ is *isolated* if there exists a neighborhood U of Λ such that $\Lambda = \cap_{t\in\mathbb{R}} X^t(U)$. A compact invariant set Λ is *attracting* if $\Lambda_X(U) := \cap_{t\geq 0} X^t(U)$ equals Λ for some neighborhood U of Λ satisfying $\overline{X^t(U)} \subset U$, for all $t > 0$. In this case the neighborhood U is called an *isolating* neighborhood of Λ. Note that $\Lambda_X(U)$ is in general different from $\cap_{t\in\mathbb{R}} X^t(U)$, but for an attracting set the extra condition $\overline{X^t(U)} \subset U$ for $t > 0$ ensures that every attracting set is also isolated. We say that Λ is *transitive* if Λ is the closure of both $\{X^t(q) : t > 0\}$ and $\{X^t(q) : t < 0\}$ for some $q \in \Lambda$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. We say that Λ is a *proper* attractor or repeller if $\emptyset \neq \Lambda \neq M$.

An *equilibrium* (or *singularity*) for X is a point $\sigma \in M$ such that $X^t(\sigma) = \sigma$ for all $t \in \mathbb{R}$, i.e. a fixed point of all the flow maps, which corresponds to a zero of the associated vector field X: $X(\sigma) = 0$. An *orbit* of X is a set $\mathcal{O}(q) = \mathcal{O}_X(q) = \{X^t(q) : t \in \mathbb{R}\}$ for some $q \in M$. A *periodic orbit* of X is an orbit $\mathcal{O} = \mathcal{O}_X(p)$ such that $X^T(p) = p$ for some minimal $T > 0$. A *critical element* of a given vector field X is either an equilibrium or a periodic orbit.

We recall that a X^t-invariant probability measure μ is a probability measure satisfying $\mu(X^t(A)) = \mu(A)$ for all $t \in \mathbb{R}$ and measurable $A \subset M$. Given an invariant probability measure μ for a flow X^t, let $B(\mu)$ be the the *(ergodic) basin* of μ, i.e., the set of points $z \in M$ satisfying for all continuous functions $\varphi : M \to \mathbb{R}$

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \varphi\big(X^t(z)\big)\, dt = \int \varphi \, d\mu.$$

We say that μ is a *physical* (or *SRB*) measure for X if $B(\mu)$ has positive Lebesgue measure: $\mathrm{Leb}\big(B(\mu)\big) > 0$.

The existence of a physical measures for an attractor shows that most points in a neighborhood of the attractor have well defined long term statistical behavior. So, in spite of chaotic behavior preventing the exact prediction of the time evolution of the system in practical terms, we gain some statistical knowledge of the long term behavior of the system near the chaotic attractor.

1.3 Chaotic systems

We distinguish between forward and backward sensitive dependence on initial conditions. We say that an invariant subset Λ for a flow X^t is *future chaotic with constant* $r > 0$ if, for every $x \in \Lambda$ and each neighborhood U of x in the ambient manifold, there exists $y \in U$ and $t > 0$ such that $\operatorname{dist}\big(X^t(y), X^t(x)\big) \geq r$. Analogously we say that Λ is *past chaotic with constant* r if Λ is future chaotic with constant r for the flow generated by $-X$. If we have such *sensitive dependence both for the past and for the future*, we say that Λ is *chaotic*. Note that in this language sensitive dependence on initial conditions is weaker than chaotic, future chaotic or past chaotic conditions.

An easy consequence of chaotic behavior is that it prevents the existence of sources or sinks, either attracting or repelling equilibria or periodic orbits, inside the invariant set Λ. Indeed, if Λ is future chaotic (for some constant $r > 0$) then, were it to contain some attracting periodic orbit or equilibrium, any point of such orbit (or equilibrium) would admit no point in a neighborhood whose orbit would move away in the future. Likewise, reversing the time direction, a past chaotic invariant set cannot contain repelling periodic orbits or repelling equilibria. As an almost reciprocal we have the following.

Lemma 1. *If $\Lambda = \cap_{t\in\mathbb{R}} X^t(U)$ is a compact isolated proper subset for $X \in \mathfrak{X}^1(M)$ with isolating neighborhood U and Λ is* not future chaotic *(respective not past chaotic), then $\Lambda_X^-(U) := \overline{\cap_{t>0} X^{-t}(U)}$ (respective $\Lambda_X^+(U) := \overline{\cap_{t>0} X^t(U)}$) has non-empty interior.*

Proof. If Λ is not future chaotic, then for every $r > 0$ there exists some point $x \in \Lambda$ and a neighborhood V of x such that $\operatorname{dist}\big(X^t(y), X^t(x)\big) < r$ for all $t > 0$ and each $y \in V$. If we choose $0 < r < \operatorname{dist}(M \setminus U, \Lambda)$ (we note that if $\Lambda = U$ then Λ would be open and closed, and so, by connectedness of M, Λ would not be a proper subset), then we deduce that $X^t(y) \in U$, that is, $y \in X^{-t}(U)$ for all $t > 0$, hence $V \subset \Lambda_X^-(U)$. Analogously if Λ is not past chaotic, just by reversing the time direction. □

In particular *if an invariant and isolated set Λ with isolating neighborhood U is given such that the volume of both $\Lambda_X^+(U)$ and $\Lambda_X^-(U)$ is zero, then Λ is chaotic*.

Sensitive dependence on initial conditions is part of the definition of *chaotic system* in the literature, see e.g. [7]. It is an interesting fact that sensitive dependence is a consequence of another two common features of most systems considered to be chaotic: existence of a dense orbit and existence of a dense subset of periodic orbits.

Proposition 1. *A compact invariant subset Λ for a flow X^t with a dense subset of periodic orbits and a dense (regular and non-periodic) orbit is chaotic.*

A short proof of this proposition can be found in [4]. An extensive discussion of this and related topics can be found in [9].

1.4 Lack of sensitiveness for flows on surfaces

We recall the following celebrated result of Mauricio Peixoto in [24, 25] (and for a more detailed exposition of this results and sketch of the proof see [11]) built on previous work of Poincaré [26] and Andronov and Pontryagin [1], that characterizes structurally stable vector fields on compact surfaces.

Theorem 1 (Peixoto). *A C^r vector field, $r \geq 1$, on a compact surface S is structurally stable if, and only if:*

1. *the number of critical elements is finite and each is hyperbolic;*
2. *there are no orbits connecting saddle points;*
3. *the non-wandering set consists of critical elements alone.*

Moreover if S is orientable, then the set of structurally stable vector fields is open and dense in $\mathfrak{X}^r(S)$.

In particular, this implies that for a structurally stable vector field X on S there is an open and dense subset B of S such that the positive orbit $X^t(p), t \geq 0$ of $p \in B$ converges to one of finitely many attracting equilibria. Therefore *no sensitive dependence on initial conditions arises for an open and dense subset of all vector fields in orientable surfaces*.

The extension of Peixoto's characterization of structural stability for C^r flows, $r \geq 1$, on non-orientable surfaces is known as *Peixoto's Conjecture*, and up until now it has been proved for the projective plane $\mathbb{P}^2$ [22], the Klein bottle $\mathbb{K}^2$ [18] and $\mathbb{L}^2$, the torus with one cross-cap [12]. Hence for these surfaces we also have no sensitiveness to initial conditions for most vector fields.

This explains in part the great interest attached to the Lorenz attractor which was one of the first examples of sensitive dependence on initial conditions.

1.5 Robustness and volume hyperbolicity

Related to chaotic behavior is the notion of *robust dynamics*. We say that an attracting set $\Lambda = \Lambda_X(U)$ for a 3-flow X and some open subset U is *robust* if there exists a C^1 neighborhood $\mathscr{U}$ of X in $\mathfrak{X}^1(M)$ such that $\Lambda_Y(U)$ is transitive for every $Y \in \mathscr{U}$.

The following result obtained by Morales, Pacifico and Pujals in [20] characterizes robust attractors for three-dimensional flows.

Theorem 2. *Robust attractors for flows containing equilibria are singular-hyperbolic sets for X.*

We remark that robust attractors cannot be C^1 approximated by vector fields presenting either attracting or repelling periodic points. This implies that, on 3-manifolds, any periodic orbit inside a robust attractor is hyperbolic of saddle-type.

We now define the concept of singular-hyperbolicity. A compact invariant set Λ of X is *partially hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ and constants $\lambda, K > 0$ such that

- E^c_Λ (K,λ)-*dominates* E^s_Λ, i.e. for all $x \in \Lambda$ and for all $t \ge 0$

$$\|DX^t(x) \mid E^s_x\| \le \frac{e^{-\lambda t}}{K} \cdot m(DX^t(x) \mid E^c_x); \tag{1.1}$$

- E^s_Λ is (K,λ)-contracting: $\|DX^t \mid E^s_x\| \le Ke^{-\lambda t}$ for all $x \in \Lambda$ and for all $t \ge 0$.

For $x \in \Lambda$ and $t \in \mathbb{R}$ we let $J^c_t(x)$ be the absolute value of the determinant of the linear map $DX^t(x) \mid E^c_x : E^c_x \to E^c_{X^t(x)}$. We say that the sub-bundle E^c_Λ of the partial hyperbolic set Λ is (K,λ)-*volume expanding* if

$$J^c_t(x) = \left|\det(DX^t \mid E^c_x)\right| \ge Ke^{\lambda t},$$

for every $x \in \Lambda$ and $t \ge 0$.

We say that a partially hyperbolic set is *singular-hyperbolic* if its singularities are hyperbolic and it has volume expanding central direction.

A *singular-hyperbolic attractor* is a singular-hyperbolic set which is an attractor as well: an example is the (geometric) Lorenz attractor presented in [16, 10]. Any equilibrium σ of a singular-hyperbolic attractor for a vector field X is such that $DX(\sigma)$ has only real eigenvalues $\lambda_2 \le \lambda_3 \le \lambda_1$ satisfying the same relations as in the Lorenz flow example:

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1, \tag{1.2}$$

which we refer to as *Lorenz-like equilibria*. We recall that an compact X^t-invariant set Λ is *hyperbolic* if the tangent bundle over Λ splits $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ into three DX^t-invariant subbundles, where E^s_Λ is uniformly contracted, E^u_Λ is uniformly expanded, and E^X_Λ is the direction of the flow at the points of Λ. It is known, see [20], that a partially hyperbolic set for a three-dimensional flow, with volume expanding central direction and without equilibria, is hyperbolic. Hence the notion of singular-hyperbolicity is an extension of the notion of hyperbolicity.

Recently in a joint work with Pacifico, Pujals and Viana [3] the following consequence of transitivity and singular-hyperbolicity was proved.

Theorem 3. *Let $\Lambda = \Lambda_X(U)$ be a singular-hyperbolic* attractor *of a flow $X \in \mathfrak{X}^2(M)$ on a three-dimensional manifold. Then Λ supports a unique physical probability measure μ which is ergodic and its ergodic basin covers a full Lebesgue measure subset of the topological basin of attraction, i.e. $B(\mu) = W^s(\Lambda)$ Lebesgue mod* 0. *Moreover the support of μ is the whole attractor Λ.*

It follows from the proof in [3] that the singular-hyperbolic *attracting set* $\Lambda_Y(U)$ for all $Y \in \mathfrak{X}^2(M)$ which are C^1-close enough to X *admits finitely many physical measures whose ergodic basins cover U except for a zero volume subset.*

1.5.1 Absence of sinks and sources nearby

The proof of Theorem 2 given in [20] uses several tools from the theory of normal hyperbolicity developed first by Mañé in [17] together with the low dimension of the flow. Indeed, going through the proof in [20] we can see that the arguments can be carried through assuming that

1. Λ is an attractor for X with isolating neighborhood U such that every equilibria in U is hyperbolic with no resonances;
2. there exists a C^1 neighborhood $\mathscr{U}$ of X such that for all $Y \in \mathscr{U}$ every periodic orbit and equilibria in U is hyperbolic of saddle-type.

The condition on the equilibria amounts to restricting the possible three-dimensional vector fields in the above statement to an open a dense subset of all C^1 vector fields. Indeed, the hyperbolic and no-resonance condition on a equilibrium σ means that:

- either $\lambda \neq \Re(\omega)$ if the eigenvalues of $DX(\sigma)$ are $\lambda \in \mathbb{R}$ and $\omega, \overline{\omega} \in \mathbb{C}$;
- or σ has only real eigenvalues with different norms.

Indeed, conditions (1) and (2) ensure that no bifurcations of periodic orbits or equilibria leading to sinks or sources are allowed for any nearby flow in U. This implies, by now standard arguments, that the flow on Λ must have a dominated splitting which is *volume hyperbolic*: both subbundles of the splitting must contract/expand volume. For a 3-dimensional flow one of the subbundles is one-dimensional, and so we deduce singular-hyperbolicity either for X or for $-X$. If Λ has no equilibria, then Λ is uniformly hyperbolic. Otherwise, it follows from the arguments in [20] that all singularities of Λ are Lorenz-like and this shows that Λ must be singular-hyperbolic for X.

We note that the second condition above is a consequence of any one of the following assumptions on U:

robust chaoticity for every $Y \in \mathscr{U}$ the maximal invariant subset $\Lambda_Y(U)$ is chaotic;
zero volume and future chaoticity for every $Y \in \mathscr{U}$ the maximal invariant subset $\Lambda_Y(U)$ has zero volume and is future chaotic;
zero volume and robust positive Lyapunov exponent for every $Y \in \mathscr{U}$ the maximal invariant subset $\Lambda_Y(U)$ has zero volume and there exists a full Lebesgue measure subset P_Y of U such that

$$\limsup \frac{1}{n} \sum_{i=0}^{n-1} \log \|DY_x^i\| > 0, \quad x \in P_Y. \tag{1.3}$$

The result of Mañé analogous to Theorem 2 in [17]

Theorem 4. *Robust attractors for surface diffeomorphisms are hyperbolic.*

also follows from the absence of sinks and sources for all C^1 close diffeomorphisms in a neighborhood of the attractor.

Extensions of these results to higher dimensions for diffeomorphisms, by Bonatti, Díaz and Pujals in [5], show that robust transitive sets always admit a volume

hyperbolic splitting of the tangent bundle. Vivier in [30] extends previous results of Doering [8] for flows, showing that a C^1 robustly transitive vector field on a compact boundaryless n-manifold, with $n \geq 3$, admits a global dominated splitting. Metzger and Morales extend the arguments in [20] to homogeneous vector fields (inducing flows allowing no bifurcation of critical elements, i.e. no modification of the index of periodic orbits or equilibria) in higher dimensions leading to the concept of 2-sectional expanding attractor in [19].

1.5.2 Robust chaoticity, volume hyperbolicity and physical measure

The preceding observations allows us to deduce that robust chaoticity is a sufficient conditions for singular-hyperbolicity of a generic attractor.

Corollary 1. *Let Λ be an attractor for $X \in \mathfrak{X}^1(M^3)$ such that every equilibrium in its trapping region is hyperbolic with no resonances. Then Λ is singular-hyperbolic if, and only if, Λ is robustly chaotic.*

This means that *if we can show that arbitrarily close orbits, in an isolating neighborhood of an attractor, are driven apart, for the future as well as for the past, by the evolution of the system, and this behavior persists for all C^1 nearby vector fields, then the attractor is singular-hyperbolic*.

To prove the necessary condition on Corollary 1 we use the concept of expansiveness for flows, and through it show that singular-hyperbolic attractors for 3-flows are robustly expansive and, as a consequence, robustly chaotic also. This is done in the last Section 1.6.

We recall the following conjecture of Viana, presented in [28]

Conjecture 1. If an attracting set $\Lambda(U)$ of smooth map/flow has a non-zero Lyapunov exponent at Lebesgue almost every point of its isolated neighborhood U (i.e. it satisfies (1.3) with $P_Y \subset U$), then it admits some physical measure.

From the preceding results and observations we can give a partial answer to this conjecture for 3-flows in the following form.

Corollary 2. *Let $\Lambda_X(U)$ be an attractor for a flow $X \in \mathfrak{X}^1(M)$ such that*

- *the divergence of X is negative in U;*
- *the equilibria in U are hyperbolic with no resonances;*
- *there exists a neighborhood $\mathscr{U}$ of X in $\mathfrak{X}^1(M)$ such that for $Y \in \mathscr{U}$ one has* (1.3) *almost everywhere in U.*

Then there exists a neighborhood $\mathscr{V} \subset \mathscr{U}$ of X in $\mathfrak{X}^1(M)$ and a dense subset $\mathscr{D} \subset \mathscr{V}$ such that

1. *$\Lambda_Y(U)$ is singular-hyperbolic for all $Y \in \mathscr{V}$;*
2. *there exists a physical measure μ_Y supported in $\Lambda_Y(U)$ for all $Y \in \mathscr{D}$.*

Indeed, item (2) above is a consequence of item (1), the denseness of $\mathfrak{X}^2(M)$ in $\mathfrak{X}^1(M)$ in the C^1 topology, together with Theorem 3 and the observation following its statement.

Item (1) above is a consequence of Corollary 1 and the observations of Section 1.5.1, noting that negative divergence on the isolating neighborhood U ensures that the volume of $\Lambda_Y(U)$ is zero for Y in a C^1 neighborhood $\mathscr{V}$ of X.

1.6 Expansive systems

A concept related to sensitiveness is that of expansiveness, which roughly means that points whose orbits are always close for all time must coincide. The concept of expansiveness for homeomorphisms plays an important role in the study of transformations. Bowen and Walters [6] gave a definition of expansiveness for flows which is now called *C-expansiveness* [14]. The basic idea of their definition is that two points which are not close in the orbit topology induced by $\mathbb{R}$ can be separated at the same time even if one allows a continuous time lag — see below for the technical definitions. The equilibria of C-expansive flows must be isolated [6, Proposition 1] which implies that the Lorenz attractors and geometric Lorenz models are not C-expansive.

Keynes and Sears introduced [14] the idea of restriction of the time lag and gave several definitions of expansiveness weaker than C-expansiveness. The notion of *K-expansiveness* is defined allowing only the time lag given by an increasing surjective homeomorphism of $\mathbb{R}$. Komuro [15] showed that the Lorenz attractor and the geometric Lorenz models are not K-expansive. The reason for this is not that the restriction of the time lag is insufficient, but that the topology induced by $\mathbb{R}$ is unsuited to measure the closeness of two points in the same orbit.

Taking this fact into consideration, Komuro [15] gave a definition of *expansiveness* suitable for flows presenting equilibria accumulated by regular orbits. This concept is enough to show that two points which do not lie on a same orbit can be separated.

Let $C\big(\mathbb{R},\mathbb{R}\big)$ be the set of all continuous functions $h:\mathbb{R}\to\mathbb{R}$ and let us write $C\big((\mathbb{R},0),(\mathbb{R},0)\big)$ for the subset of all $h\in C\big(\mathbb{R},\mathbb{R}\big)$ such that $h(0)=0$. We define

$$\mathscr{K}_0=\{h\in C\big((\mathbb{R},0),(\mathbb{R},0)\big):h(\mathbb{R})=\mathbb{R},\,h(s)>h(t)\,,\forall s>t\},$$

and

$$\mathscr{K}=\{h\in C\big(\mathbb{R},\mathbb{R}\big):h(\mathbb{R})=\mathbb{R},\,h(s)>h(t)\,,\forall s>t\},$$

A flow X is *C-expansive* (*K-expansive* respectively) on an invariant subset $\Lambda\subset M$ if for every $\varepsilon>0$ there exists $\delta>0$ such that if $x,y\in\Lambda$ and for some $h\in\mathscr{K}_0$ (respectively $h\in\mathscr{K}$) we have

$$\operatorname{dist}\big(X^t(x), X^{h(t)}(y)\big) \le \delta \quad \text{for all} \quad t \in \mathbb{R}, \tag{1.4}$$

then $y \in X^{[-\varepsilon,\varepsilon]}(x) = \{X^t(x) : -\varepsilon \le t \le \varepsilon\}$.

We say that the flow X is *expansive* on Λ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in \Lambda$ and some $h \in \mathscr{K}$ (note that now we do not demand that 0 be fixed by h) satisfying (1.4), then we can find $t_0 \in \mathbb{R}$ such that $X^{h(t_0)}(y) \in X^{[t_0-\varepsilon, t_0+\varepsilon]}(x)$.

Observe that expansiveness on M implies sensitive dependence on initial conditions for any flow on a manifold with dimension at least 2. Indeed if ε, δ satisfy the expansiveness condition above with h equal to the identity and we are given a point $x \in M$ and a neighborhood U of x, then taking $y \in U \setminus X^{[-\varepsilon,\varepsilon]}(x)$ (which always exists since we assume that M is not one-dimensional) there exists $t \in \mathbb{R}$ such that $\operatorname{dist}\big(X^t(y), X^t(x)\big) \ge \delta$. The same argument applies whenever we have expansiveness on an X-invariant subset Λ of M containing a dense regular orbit of the flow.

Clearly C-expansive $\Longrightarrow$ K-expansive $\Longrightarrow$ expansive by definition. When a flow has no fixed point then C-expansiveness is equivalent to K-expansiveness [21, Theorem A]. In [6] it is shown that on a connected manifold a C-expansive flow has no fixed points. The following was kindly communicated to us by Alfonso Artigue from IMERL, the proof can be found in [2].

Proposition 2. *A flow is C-expansive on a manifold M if, and only if, it is K-expansive.*

We will see that singular-hyperbolic attractors are expansive. In particular, the Lorenz attractor and the geometric Lorenz examples are all expansive and sensitive to initial conditions. Since these families of flows exhibit equilibria accumulated by regular orbits, we see that expansiveness is compatible with the existence of fixed points by the flow.

1.6.1 Singular-hyperbolicity and expansiveness

The full proof of the following given in [3].

Theorem 5. *Let Λ be a singular-hyperbolic attractor of $X \in \mathfrak{X}^1(M)$. Then Λ is expansive.*

The reasoning is based on analyzing Poincaré return maps of the flow to a convenient (δ-adapted) cross-section. We use the family of adapted cross-sections and corresponding Poincará maps R, whose Poincaré time $t(\cdot)$ is large enough, obtained assuming that the attractor Λ is singular-hyperbolic. These cross-sections have a codimension 1 foliation, which are dynamically defined, whose leaves are uniformly contracted and invariant under the Poincaré maps. In addition R is uniformly expanding in the transverse direction and this also holds near the singularities.

From here we argue by contradiction: if the flow is not expansive on Λ, then we can find a pair of orbits hitting the cross-sections infinitely often on pairs of points

uniformly close. We derive a contradiction by showing that the uniform expansion in the transverse direction to the stable foliation must take the pairs of points apart, unless one orbit is on the stable manifold of the other.

This argument only depends on the existence of finitely many Lorenz-like singularities on a compact partially hyperbolic invariant attracting subset $\Lambda = \Lambda_X(U)$, with volume expanding central direction, and of a family of adapted cross-sections with Poincaré maps between them, whose derivative is hyperbolic. It is straightforward that if these conditions are satisfied for a flow X^t of $X \in \mathfrak{X}^1(M^3)$, then the same conditions hold for all C^1 nearby flows Y^t and for the maximal invariant subset $\Lambda_Y(U)$ *with the same family of cross-sections* which are also adapted to $\Lambda_Y(U)$ (as long as Y is C^1-close enough to X).

Corollary 3. *A singular-hyperbolic attractor $\Lambda = \Lambda_X(U)$ is* robustly expansive, *that is, there exists a neighborhood $\mathscr{U}$ of X in $\mathfrak{X}^1(M)$ such that $\Lambda_Y(U)$ is expansive for each $Y \in \mathscr{U}$, where U is an isolating neighborhood of Λ.*

Indeed, since transversality, partial hyperbolicity and volume expanding central direction are robust properties, and also the hyperbolicity of the Poincaré maps depends on the central volume expansion, all we need to do is to check that a given adapted cross-section Σ to X is also adapted to $Y \in \mathfrak{X}^1$ for every Y sufficiently C^1 close to X. But $\Lambda_X(U)$ and $\Lambda_Y(U)$ are close in the Hausdorff distance if X and Y are close in the C^0 distance, by the following elementary result.

Lemma 2. *Let Λ be an isolated set of $X \in \mathfrak{X}^r(M)$, $r \geq 0$. Then for every isolating block U of Λ and every $\varepsilon > 0$ there is a neighborhood $\mathscr{U}$ of X in $\mathfrak{X}^r(M)$ such that $\Lambda_Y(U) \subset B(\Lambda,\varepsilon)$ and $\Lambda \subset B(\Lambda_Y(U),\varepsilon)$ for all $Y \in \mathscr{U}$.*

Thus, if Σ is an adapted cross-section we can find a C^1-neighborhood $\mathscr{U}$ of X in $\mathfrak{X}^1$ such that Σ is still adapted to every flow Y^t generated by a vector field in $\mathscr{U}$.

1.6.2 Adapted cross-sections and Poincaré maps

To help explain the ideas of the proofs we give here a few properties of *Poincaré maps*, that is, continuous maps $R : \Sigma \to \Sigma'$ of the form $R(x) = X^{t(x)}(x)$ between cross-sections Σ and Σ' of the flow near a singular-hyperbolic set. We always assume that the Poincaré time $t(\cdot)$ is large enough as explained in what follows.

We assume that Λ is a compact invariant subset for a flow $X \in \mathfrak{X}^1(M)$ such that Λ is a singular-hyperbolic attractor. Then every equilibrium in Λ is Lorenz-like.

1.6.2.1 Stable foliations on cross-sections

We start recalling standard facts about uniformly hyperbolic flows from e.g. [13].

An embedded disk $\gamma \subset M$ is a (local) *strong-unstable manifold*, or a *strong-unstable disk*, if $\operatorname{dist}(X^{-t}(x), X^{-t}(y))$ tends to zero exponentially fast as $t \to +\infty$,

for every $x,y \in \gamma$. Similarly, γ is called a (local) *strong-stable manifold*, or a *strong-stable disk*, if $\mathrm{dist}(X^t(x),X^t(y)) \to 0$ exponentially fast as $n \to +\infty$, for every $x,y \in \gamma$. It is well-known that every point in a uniformly hyperbolic set possesses a local strong-stable manifold $W^{ss}_{loc}(x)$ and a local strong-unstable manifold $W^{uu}_{loc}(x)$ which are disks tangent to E_x and G_x at x respectively with topological dimensions $d_E = \dim(E)$ and $d_G = \dim(G)$ respectively. Considering the action of the flow we get the (global) *strong-stable manifold*

$$W^{ss}(x) = \bigcup_{t>0} X^{-t}\Big(W^{ss}_{loc}\big(X^t(x)\big)\Big)$$

and the (global) *strong-unstable manifold*

$$W^{uu}(x) = \bigcup_{t>0} X^{t}\Big(W^{uu}_{loc}\big(X_{-t}(x)\big)\Big)$$

for every point x of a uniformly hyperbolic set. These are immersed submanifolds with the same differentiability of the flow. We also consider the *stable manifold* $W^s(x) = \cup_{t\in\mathbb{R}} X^t\big(W^{ss}(x)\big)$ and *unstable manifold* $W^u(x) = \cup_{t\in\mathbb{R}} X^t\big(W^{uu}(x)\big)$ for x in a uniformly hyperbolic set, which are flow invariant.

Now we recall classical facts about partially hyperbolic systems, especially existence of strong-stable and center-unstable foliations. The standard reference is [13].

We have that Λ is a singular-hyperbolic isolated set of $X \in \mathfrak{X}^1(M)$ with invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ with $\dim E^{cu} = 2$. Let $\tilde{E}^s \oplus \tilde{E}^{cu}$ be a continuous extension of this splitting to a small neighborhood U_0 of Λ. For convenience we take U_0 to be forward invariant. Then $\tilde{E}^s$ may be chosen invariant under the derivative: just consider at each point the direction formed by those vectors which are strongly contracted by DX^t for positive t. In general $\tilde{E}^{cu}$ is not invariant. However we can consider a cone field around it on U_0

$$C^{cu}_a(x) = \{v = v^s + v^{cu} : v^s \in \tilde{E}^s_x \text{ and } v^{cu} \in \tilde{E}^{cu}_x \text{ with } \|v^s\| \le a \cdot \|v^{cu}\|\}$$

which is forward invariant for $a > 0$:

$$DX^t(C^{cu}_a(x)) \subset C^{cu}_a(X^t(x)) \quad \text{for all large } t > 0. \tag{1.5}$$

Moreover we may take $a > 0$ arbitrarily small, reducing U_0 if necessary. For notational simplicity we write E^s and E^{cu} for $\tilde{E}^s$ and $\tilde{E}^{cu}$ in all that follows.

From the standard normal hyperbolic theory, there are locally strong-stable and center-unstable manifolds, defined at every regular point $x \in U_0$ and which are embedded disks tangent to $E^s(x)$ and $E^{cu}(x)$, respectively. Given any $x \in U_0$ define the strong-stable manifold $W^{ss}(x)$ and the stable-manifold $W^s(x)$ as for an hyperbolic flow (see the beginning of this section).

Denoting $E^{cs}_x = E^s_x \oplus E^X_x$, where E^X_x is the direction of the flow at x, it follows that

$$T_x W^{ss}(x) = E^s_x \quad \text{and} \quad T_x W^s(x) = E^{cs}_x.$$

We fix ε once and for all. Then we call $W^{ss}_\varepsilon(x)$ the local *strong-stable manifold* and $W^{cu}_\varepsilon(x)$ the local *center-unstable manifold* of x.

Now let Σ be a *cross-section* to the flow, that is, a C^1 embedded compact disk transverse to X: at every point $z \in \Sigma$ we have $T_z\Sigma \oplus E^X_z = T_zM$ (recall that E^X_z is the one-dimensional subspace $\{s \cdot X(z) : s \in \mathbb{R}\}$). For every $x \in \Sigma$ we define $W^s(x,\Sigma)$ to be the connected component of $W^s(x) \cap \Sigma$ that contains x. This defines a foliation $\mathscr{F}^s_\Sigma$ of Σ into co-dimension 1 sub-manifolds of class C^1.

Given any cross-section Σ and a point x in its interior, we may always find a smaller cross-section also with x in its interior and which is the image of the square $[0,1] \times [0,1]$ by a C^1 diffeomorphism h that sends horizontal lines inside leaves of $\mathscr{F}^s_\Sigma$. In what follows we assume that cross-sections are of this kind, see Figure 1.2.

We also assume that each cross-section Σ is contained in U_0, so that every $x \in \Sigma$ is such that $\omega(x) \subset \Lambda$.

On the one hand $x \mapsto W^{ss}_\varepsilon(x)$ is usually not differentiable if we assume that X is only of class C^1, see e.g. [23]. On the other hand, assuming that the cross-section is small with respect to ε, and choosing any curve $\gamma \subset \Sigma$ crossing transversely every leaf of $\mathscr{F}^s_\Sigma$, we may consider a Poincaré map

$$R_\Sigma : \Sigma \to \Sigma(\gamma) = \bigcup_{z \in \gamma} W^{ss}_\varepsilon(z)$$

with Poincaré time close to zero, see Figure 1.2. This is a homeomorphism onto its image, close to the identity, such that $R_\Sigma(W^s(x,\Sigma)) \subset W^{ss}_\varepsilon(R_\Sigma(x))$. So, identifying the points of Σ with their images under this homeomorphism, we pretend that indeed $W^s(x,\Sigma) \subset W^{ss}_\varepsilon(x)$.

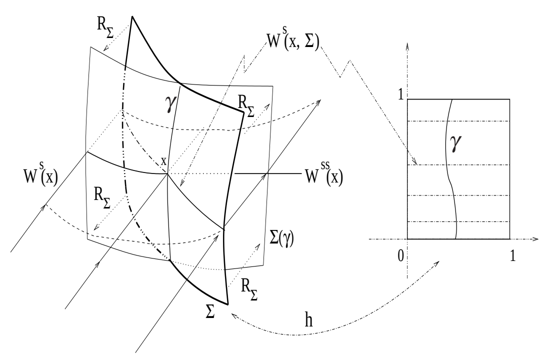

Fig. 1.2 The sections Σ, $\Sigma(\gamma)$, the manifolds $W^s(x), W^{ss}(x)$, $W^s(x,\Sigma)$ and the projection R_Σ, on the right. On the left, the square $[0,1] \times [0,1]$ is identified with Σ through the map h, where $\mathscr{F}^s_\Sigma$ becomes the horizontal foliation and the curve γ is transverse to the horizontal direction. Solid lines with arrows indicate the flow direction.

1.6.2.2 Hyperbolicity of Poincaré maps

Let Σ be a small cross-section to X and let $R : \Sigma \to \Sigma'$ be a Poincaré map $R(y) = X^{t(y)}(y)$ to another cross-section Σ' (possibly $\Sigma = \Sigma'$). Here R needs not correspond to the first time the orbits of Σ encounter Σ'.

The splitting $E^s \oplus E^{cu}$ over U_0 induces a continuous splitting $E^s_\Sigma \oplus E^{cu}_\Sigma$ of the tangent bundle $T\Sigma$ to Σ (and analogously for Σ'), defined by

$$E^s_\Sigma(y) = E^{cs}_y \cap T_y\Sigma \quad \text{and} \quad E^{cu}_\Sigma(y) = E^{cu}_y \cap T_y\Sigma. \tag{1.6}$$

We now show that if the Poincaré time $t(x)$ is sufficiently large then (1.6) defines a hyperbolic splitting for the transformation R on the cross-sections restricted to Λ.

Proposition 3. *Let $R : \Sigma \to \Sigma'$ be a Poincaré map as before with Poincaré time $t(\cdot)$. Then $DR_x(E^s_\Sigma(x)) = E^s_{\Sigma'}(R(x))$ at every $x \in \Sigma$ and $DR_x(E^{cu}_\Sigma(x)) = E^{cu}_{\Sigma'}(R(x))$ at every $x \in \Lambda \cap \Sigma$.*

Moreover for every given $0 < \lambda < 1$ there exists $T_1 = T_1(\Sigma, \Sigma', \lambda) > 0$ such that if $t(\cdot) > T_1$ at every point, then

$$\|DR \mid E^s_\Sigma(x)\| < \lambda \quad \textit{and} \quad \|DR \mid E^{cu}_\Sigma(x)\| > 1/\lambda \quad \textit{at every } x \in \Sigma.$$

Given a cross-section Σ, a positive number ρ, and a point $x \in \Sigma$, we define the unstable cone of width ρ at x by

$$C^u_\rho(x) = \{v = v^s + v^u : v^s \in E^s_\Sigma(x),\, v^u \in E^{cu}_\Sigma(x) \text{ and } \|v^s\| \le \rho \|v^u\|\}. \tag{1.7}$$

Let $\rho > 0$ be any small constant. In the following consequence of Proposition 3 we assume the neighborhood U_0 has been chose sufficiently small.

Corollary 4. *For any $R : \Sigma \to \Sigma'$ as in Proposition 3, with $t(\cdot) > T_1$, and any $x \in \Sigma$, we have $DR(x)(C^u_\rho(x)) \subset C^u_{\rho/2}(R(x))$ and*

$$\|DR_x(v)\| \ge \frac{5}{6}\lambda^{-1} \cdot \|v\| \quad \textit{for all} \quad v \in C^u_\rho(x).$$

As usual a *curve* is the image of a compact interval $[a,b]$ by a C^1 map. We use $\ell(\gamma)$ to denote its length. By a *cu-curve* in Σ we mean a curve contained in the cross-section Σ and whose tangent direction is contained in the unstable cone $T_z\gamma \subset C^u_\rho(z)$ for all $z \in \gamma$. The next lemma says that *the length of cu-curves linking the stable leaves of nearby points x,y must be bounded by the distance between x and y.*

Lemma 3. *Let us we assume that ρ has been fixed, sufficiently small. Then there exists a constant κ such that, for any pair of points $x, y \in \Sigma$, and any cu-curve γ joining x to some point of $W^s(y,\Sigma)$, we have $\ell(\gamma) \le \kappa \cdot d(x,y)$.*

Here d is the intrinsic distance in the C^2 surface Σ, that is, the length of the shortest smooth curve inside Σ connecting two given points in Σ.

In what follows we take T_1 in Proposition 3 for $\lambda = 1/3$.

1.6.2.3 Adapted cross-sections

Now we exhibit stable manifolds for Poincaré transformations $R : \Sigma \to \Sigma'$. The natural candidates are the intersections $W^s(x,\Sigma) = W^s_\varepsilon(x) \cap \Sigma$ we introduced previ-

ously. These intersections are tangent to the corresponding sub-bundle E^s_Σ and so, by Proposition 3, they are contracted by the transformation. For our purposes it is also important that the stable foliation be invariant:

$$R(W^s(x,\Sigma)) \subset W^s(R(x),\Sigma') \qquad \text{for every } x \in \Lambda \cap \Sigma. \tag{1.8}$$

In order to have this we restrict our class of cross-sections whose center-unstable boundary is disjoint from Λ. Recall that we are considering cross-sections Σ that are diffeomorphic to the square $[0,1]\times[0,1]$, with the horizontal lines $[0,1]\times\{\eta\}$ being mapped to stable sets $W^s(y,\Sigma)$. The *stable boundary* $\partial^s\Sigma$ is the image of $[0,1]\times\{0,1\}$. The *center-unstable boundary* $\partial^{cu}\Sigma$ is the image of $\{0,1\}\times[0,1]$. The cross-section is δ*-adapted* if

$$d(\Lambda\cap\Sigma,\partial^{cu}\Sigma) > \delta,$$

where d is the intrinsic distance in Σ, see Figure 1.3.

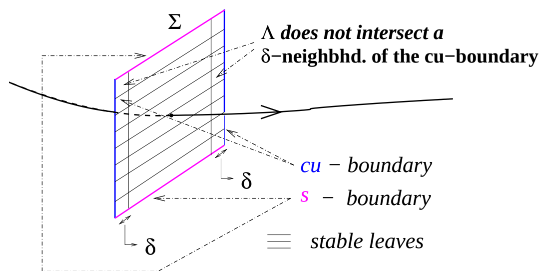

Fig. 1.3 An adapted cross-section for Λ.

Lemma 4. *Let $x \in \Lambda$ be a regular point, that is, such that $X(x) \neq 0$. There exists $\delta > 0$ such that there exists a δ-adapted cross-section Σ at x.*

We are going to show that if the cross-sections are adapted, then we have the invariance property (1.8). Given $\Sigma,\Sigma' \in \Xi$ we set $\Sigma(\Sigma') = \{x \in \Sigma : R(x) \in \Sigma'\}$ the domain of the return map from Σ to Σ'.

Lemma 5. *Given $\delta > 0$ and δ-adapted cross-sections Σ and Σ', there exists $T_2 = T_2(\Sigma,\Sigma') > 0$ such that if $R : \Sigma(\Sigma') \to \Sigma'$ defined by $R(z) = R_{t(z)}(z)$ is a Poincaré map with time $t(\cdot) > T_2$, then*

1. *$R\big(W^s(x,\Sigma)\big) \subset W^s(R(x),\Sigma')$ for every $x \in \Sigma(\Sigma')$, and also*
2. *$d(R(y),R(z)) \le \frac{1}{2} d(y,z)$ for every $y,z \in W^s(x,\Sigma)$ and $x \in \Sigma(\Sigma')$.*

Clearly we may choose $T_2 > T_1$ so that all the properties of the Poincaré maps obtained up to here are valid for return times greater than T_2.

1.6.3 Sketch of the proof of expansiveness

Here we sketch the proof of Theorem 5. The proof is by contradiction: let us suppose that there exist $\varepsilon > 0$, a sequence $\delta_n \to 0$, a sequence of functions $h_n \in \mathscr{K}$ and sequences of points $x_n, y_n \in \Lambda$ such that

$$d\big(X^t(x_n), X^{h_n(t)}(y_n)\big) \le \delta_n \quad \text{for all } t \in \mathbb{R}, \tag{1.9}$$

but

$$X^{h_n(t)}(y_n) \notin X^{[t-\varepsilon, t+\varepsilon]}(x_n) \quad \text{for all } t \in \mathbb{R}. \tag{1.10}$$

The main step of the proof is a reduction to a forward expansiveness statement about Poincaré maps which we state in Theorem 6 below.

We are going to use the following observation: there exists some regular (i.e. non-equilibrium) point $z \in \Lambda$ which is accumulated by the sequence of ω-limit sets $\omega(x_n)$. To see that this is so, start by observing that accumulation points do exist, since Λ is compact. Moreover, if the ω-limit sets accumulate on a singularity then they also accumulate on at least one of the corresponding unstable branches which, of course, consists of regular points. We fix such a z once and for all. Replacing our sequences by subsequences, if necessary, we may suppose that for every n there exists $z_n \in \omega(x_n)$ such that $z_n \to z$.

Let Σ be a δ-adapted cross-section at z, for some small δ. Reducing δ (but keeping the same cross-section) we may ensure that z is in the interior of the subset

$$\Sigma_\delta = \{y \in \Sigma : d(y, \partial\Sigma) > \delta\}.$$

By definition, x_n returns infinitely often to the neighborhood of z_n which, on its turn, is close to z. Thus dropping a finite number of terms in our sequences if necessary, we have that the orbit of x_n intersects Σ_δ infinitely many times. Let t_n be the time corresponding to the nth intersection.

Replacing x_n, y_n, t, and h_n by $x^{(n)} = X^{t_n}(x_n)$, $y^{(n)} = X^{h_n(t_n)}(y_n)$, $t' = t - t_n$, and $h_n'(t') = h_n(t' + t_n) - h_n(t_n)$, we may suppose that $x^{(n)} \in \Sigma_\delta$, while preserving both relations (1.9) and (1.10). Moreover there exists a sequence $\tau_{n,j}$, $j \ge 0$ with $\tau_{n,0} = 0$ such that

$$x^{(n)}(j) = X^{\tau_{n,j}}(x^{(n)}) \in \Sigma_\delta \quad \text{and} \quad \tau_{n,j} - \tau_{n,j-1} > \max\{t_1, t_2\} \tag{1.11}$$

for all $j \ge 1$, where t_1 is given by Proposition 3 and t_2 is given by Lemma 5.

Theorem 6. *Given $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that if $x \in \Sigma_\delta$ and $y \in \Lambda$ satisfy*

(a) *there exist τ_j such that*

$$x_j = X^{\tau_j}(x) \in \Sigma_\delta \quad \textit{and} \quad \tau_j - \tau_{j-1} > \max\{t_1, t_2\} \quad \textit{for all } j \ge 1;$$

(b) $\operatorname{dist}\big(X^t(x), X^{h(t)}(y)\big) < \delta_0$*, for all $t > 0$ and some $h \in \mathscr{K}$;*

then there exists $s = \tau_j \in \mathbb{R}$ for some $j \ge 1$ such that $X^{h(s)}(y) \in W^{ss}_{\varepsilon_0}(X^{[s-\varepsilon_0, s+\varepsilon_0]}(x))$.

The proof of Theorem 6 will not be given here, and can be found in [3]. We explain why this implies Theorem 5. We are going to use the following observation.

Lemma 6. *There exist $\rho > 0$ small and $c > 0$, depending only on the flow, such that if z_1, z_2, z_3 are points in Λ satisfying $z_3 \in X^{[-\rho,\rho]}(z_2)$ and $z_2 \in W^{ss}_\rho(z_1)$, with z_1 away from any equilibria of X, then*

$$\operatorname{dist}(z_1, z_3) \geq c \cdot \max\{\operatorname{dist}(z_1, z_2), \operatorname{dist}(z_2, z_3)\}.$$

This is a direct consequence of the fact that the angle between E^{ss} and the flow direction is bounded from zero which, on its turn, follows from the fact that the latter is contained in the center-unstable sub-bundle E^{cu}.

We fix $\varepsilon_0 = \varepsilon$ as in (1.10) and then consider δ_0 as given by Theorem 6. Next, we fix n such that $\delta_n < \delta_0$ and $\delta_n < c\rho$, and apply Theorem 6 to $x = x^{(n)}$ and $y = y^{(n)}$ and $h = h_n$. Hypothesis (a) in the theorem corresponds to (1.11) and, with these choices, hypothesis (b) follows from (1.9). Therefore we obtain that $X^{h(s)}(y) \in W^{ss}_\varepsilon(X^{[s-\varepsilon,s+\varepsilon]}(x))$. Equivalently there is $|\tau| \leq \varepsilon$ such that $X^{h(s)}(y) \in W^{ss}_\varepsilon(X^{s+\tau}(x))$. Condition (1.10) then implies that $X^{h(s)}(y) \neq X^{s+\tau}(x)$. Hence since strong-stable manifolds are expanded under backward iteration, there exists $\theta > 0$ maximum such that

$$X^{h(s)-t}(y) \in W^{ss}_\rho(X^{s+\tau-t}(x)) \quad \text{and} \quad X^{h(s+\tau-t)}(y) \in X^{[-\rho,\rho]}(X^{h(s)-t}(y))$$

for all $0 \leq t \leq \theta$, see Figure 1.4. Moreover $s = \tau_j$ for some $j \geq 1$ so that x is close to cross-section of the flow which we can assume is uniformly bounded away from the equilibria, and then we can assume that $\|X(X^t(x))\| \geq c$ for $0 \leq t \leq \theta$. Since θ is maximum

$$\text{either } \operatorname{dist}\left(X^{h(s)-t}(y), X^{s+\tau-t}(x)\right) \geq \rho$$
$$\text{or } \operatorname{dist}\left(X^{h(s+\tau-t)}(y), X^{h(s)-t}(y)\right) \geq c_0\rho$$

for $t = \theta$, because $\|X(X^t(x))\| \geq c_0 > 0$ for $0 \leq t \leq \theta$. Using Lemma 6, we conclude

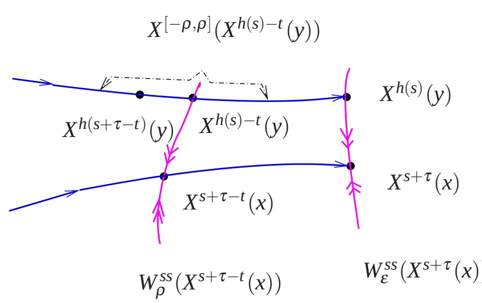

Fig. 1.4 Relative positions of the strong-stable manifolds and orbits.

that $\operatorname{dist}(X^{s+\tau-t}(x), X^{h(s+\tau-t)}(y)) \geq c\rho > \delta_n$ which contradicts (1.9). This contradiction reduces the proof of Theorem 5 to that of Theorem 6.

1.6.4 Singular-hyperbolicity and chaotic behavior

We already know that expansiveness implies sensitive dependence on initial conditions. An argument with the same flavor as the proof of expansiveness provides the following.

Theorem 7. *A singular-hyperbolic isolated set* $\Lambda = \cap_{t\in\mathbb{R}}\overline{X^t(U)}$ *is* robustly chaotic, *i.e. there exists a neighborhood* $\mathscr{U}$ *of* X *in* $\mathfrak{X}^1(M)$ *such that* $\cap_{t\in\mathbb{R}}\overline{Y^t(U)}$ *is chaotic for each* $Y\in\mathscr{U}$*, where* U *is an isolating neighborhood of* Λ*.*

Proof. The assumption of singular-hyperbolicity on an isolated proper subset Λ with isolating neighborhood U ensures that the maximal invariant subsets $\cap_{t\in\mathbb{R}}\overline{Y^t(U)}$ for all C^1 nearby flows Y are also singular-hyperbolic. Therefore to deduce robust chaotic behavior in this setting it is enough to show that a proper isolated invariant compact singular-hyperbolic subset is chaotic.

Let Λ be a singular-hyperbolic isolated proper subset for a C^1 flow. Then there exists a strong-stable manifold $W^{ss}(x)$ through each of its points x. We claim that this implies that Λ is past chaotic. Indeed, assume by contradiction that we can find $y\in W^{ss}(x)$ such that $y\neq x$ and $\operatorname{dist}\big(X^{-t}(y),X^{-t}(x)\big)<\varepsilon$ for every $t>0$, for some small $\varepsilon>0$. Then, because $W^{ss}(x)$ is uniformly contracted by the flow in positive time, there exists $\lambda>0$ such that

$$\operatorname{dist}(y,x)\le Const\cdot e^{-\lambda t}\operatorname{dist}\big(X^{-t}(y),X^{-t}(x)\big)\le Const\cdot\varepsilon e^{-\lambda t}$$

for all $t>0$, a contradiction since $y\neq x$. Hence for any given small $\varepsilon>0$ we can always find a point y arbitrarily close to x (it is enough to choose y is the strong-stable manifold of x) such that its past orbit separates from the orbit of x.

To obtain future chaotic behavior, we argue by contradiction: we assume that Λ is not future chaotic. Then for every $\varepsilon>0$ we can find a point $x\in\Lambda$ and an open neighborhood V of x such that the future orbit of each $y\in V$ is ε-close to the future orbit of x, that is, $\operatorname{dist}\big(X^t(y),X^t(x)\big)\le\varepsilon$ for all $t>0$.

First, x is not a singularity, because all the possible singularities inside a singular-hyperbolic set are hyperbolic saddles and so each singularity has a unstable manifold. Likewise, x cannot be in the stable manifold of a singularity. Therefore $\omega(x)$ contains some regular point z. Let Σ be a transversal section to the flow X^t at z.

Hence there are infinitely many times $t_n\to+\infty$ such that $x_n:=X^{t_n}(x)\in\Sigma$ and $x_n\to z$ when $n\to+\infty$. Taking Σ sufficiently small looking only to very large times, the assumption on V ensures that each $y\in V$ admits also an infinite sequence $t_n(y)\xrightarrow[n\to+\infty]{}+\infty$ satisfying

$$y_n:=X^{t_n(y)}(y)\in\Sigma\quad\text{and}\quad\operatorname{dist}(y_n,x_n)\le 10\varepsilon.$$

We can assume that $y\in V$ does not belong to $W^s(x)$, since $W^s(x)$ is a C^1 immersed sub-manifold of M. Hence we consider the connected components $\gamma_n:=W^s(x_n,\Sigma)$ and $\xi_n:=W^s(y_n,\Sigma)$ of $W^s(x)\cap\Sigma$ and $W^s(y)\cap\Sigma$, respectively. We recall that we can assume that every y in a small neighborhood of Λ admits an invariant stable

manifold because we can extend the invariant stable cone fields from Λ to a small neighborhood of Λ. We can also extend the invariant center-unstable cone fields from Λ to this same neighborhood, so that we can also define the notion of *cu*-curve in Σ in this setting.

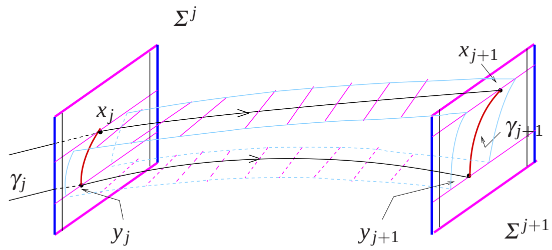

Fig. 1.5 Expansion between visits to a cross-section.

The assumption on V ensures that there exists a *cu*-curve ζ_n in Σ connecting γ_n to ξ_n, because $X^{t_n}(V)\cap\Sigma$ is an open neighborhood of x_n containing y_n. But we can assume without loss of generality that $t_{n+1}-t_n > \max\{t_1,t_2\}$, forgetting some returns to Σ in between if necessary and relabeling the times t_n. Thus Proposition 3 applies and the Poincaré return maps associated to the returns to Σ considered above are hyperbolic.

The same argument as in the proof of expansiveness guarantees that there exists a flow box connecting $\{x_n,y_n\}$ to $\{x_{n+1},y_{n+1}\}$ and sending ζ_n into a *cu*-curve $R(\zeta_n)$ connecting γ_{n+1} and ξ_{n+1}, for every $n\geq 1$.

The hyperbolicity of the Poincaré return maps ensures that the length of $R(\zeta_n)$ grows by a factor greater than one, see Figure 1.5. Therefore, since y_n,x_n are uniformly close, this implies that the length of ζ_1 and the distance between γ_1 and ξ_1 must be zero. This contradicts the choice of $y\neq W^s(x)$.

This contradiction shows that Λ is future chaotic, and concludes the proof. □

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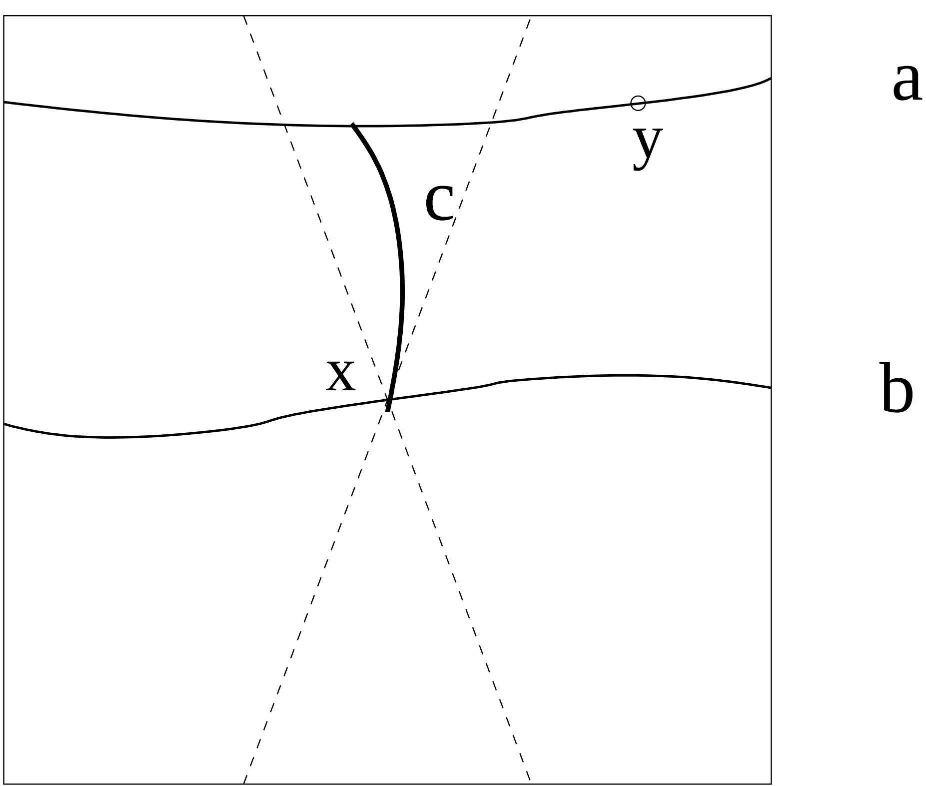
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